Conformal field theory approach to quantum Hall physics

Zlatko Papic
Institute of Physics, Belgrade,
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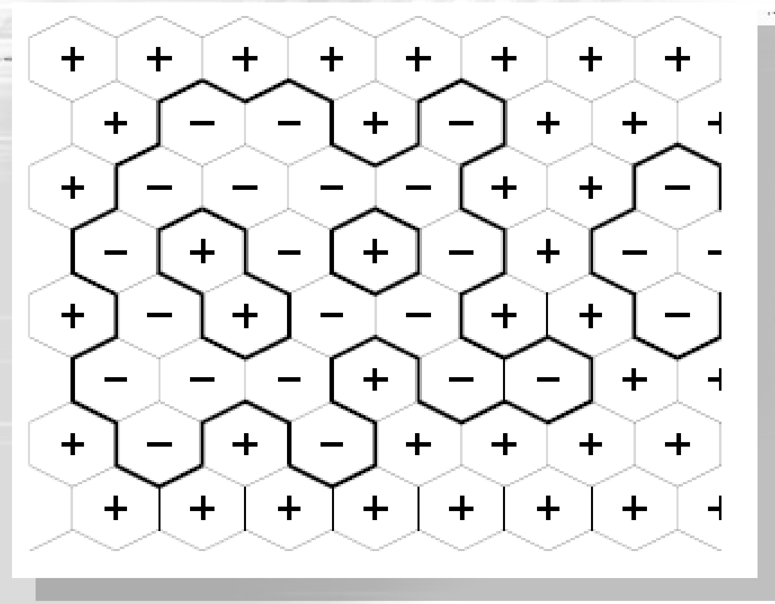
Outline

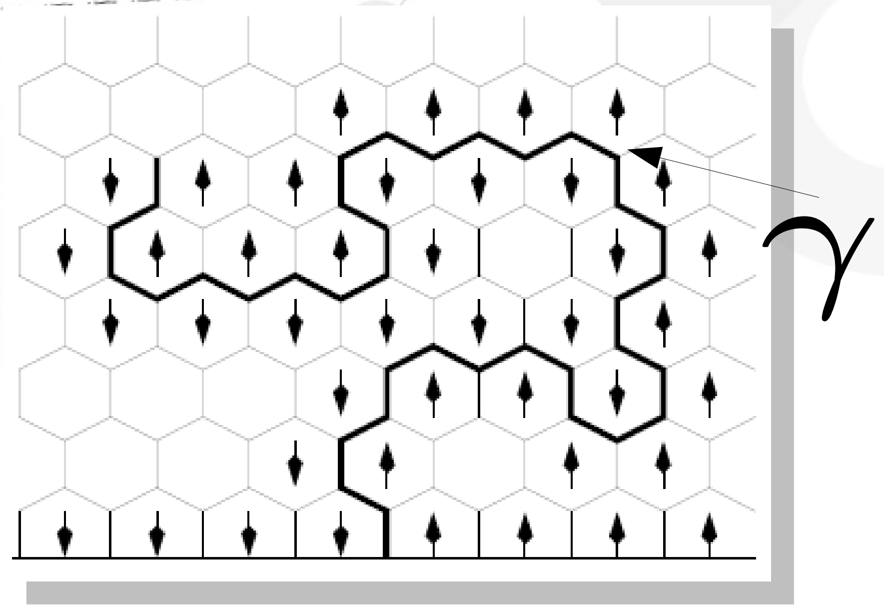
- Brief historical overview, field theory approach (algebraic) and SLE (geometrical)
- Global conformal invariance and 2D conformal invariance
- Operator formalism, radial quantization, minimal models
- CFT – QHE connection
- Entanglement entropy and entanglement spectrum for QHE states

Historical overview

- Early days of critical phenomena – how to treat them analytically?
- 1940s Onsager's solution
- 1960s RG: in the scaling limit classical critical systems = renormalizable QFTs in Euclidean spacetime
- Lattice gauge theory and CFT
- Belavin, Polyakov & Zamolodchikov paper
- Field theory <------> SLE
(local operators) (random curves)

Lattice models and the scaling limit





$$Z = Tr e^{\beta J \sum_{rr'} s(r)s(r')}$$

$$\sim Tr \prod_{rr'} (1 + x s(r) s(r'))$$

Correlation functions ->
local operators in (massless) QFT

Loop model

loops=domain walls
in dual lattice

$$< s(r_1) s(r_2) >$$

SLE= continuum limit
of γ

The Scaling Limit (1)

$\phi_j^{lat}(r)$ local observable

$$< \phi_1^{lat}(r_1)...\phi_n^{lat}(r_n) >= \frac{1}{Z}\sum_{\{s\}} \phi_1^{lat}(r_1)...\phi_n^{lat}(r_n)W(\{s\})$$

Scaling limit = take $a \to$. while keeping ξ fixed

In general, corr functions do not possess this limit. But RG and exact models suggest that particular linear combinations of local lattice observables are multiplicatively renormalisable:

$$\lim_{a\to 0} a^{-\sum_{j=1}^{n} x_j} < \phi_1^{lat}(r1)...\phi_n^{lat}(r_n) >=< \phi_1(r_1)...\phi_n(r_n) >$$

scaling fields

for certain $\{x_j\}$

scaling dimensions

The Scaling Limit (2)

Limit exists only when points $\{r_j\}$ do not coincide

The nature of singularities is given by OPE:

$$< \phi_i(r_i)\phi_j(r_j)... >= \sum_k C_{ijk}(r_i - r_j) < \phi(\frac{r_i+r_j}{2})... >$$

when $|r_i - r_j|$ is << than separation between r_i and all the other arguments in ...., C_{ijk} are independent of what is in the dots.

In the scaling limit: $\phi_j(br) = b^{-x_j}\phi_j(r)$

In massless QFT, OPE coefficients are universal:

$$C_{ijk}(r_i - r_j) = \frac{c_{ijk}}{|r_i - r_j|^{x_i+x_j-x_k}}$$

← pure numbers!

From scale to conformal invariance

suppose b varies with r – when does the following hold?

$$< \phi_1(r'_1)...\phi_n(r'_n) >_{\mathcal{D}'} = \prod_{j=1}^{n} b(r_j)^{-x_j} < \phi_1(r_1)...\phi_n(r_n) >_{\mathcal{D}}$$

$$b(r) = |\partial r' / \partial r|$$

If theory is local and the transformations $r \to r'$ look *locally* like a scale transformation + maybe rotation, it may hold. In fact, it will hold only for special kinds of fields – the *primary fields*

Transformations which are locally equivalent to a scale transformation and rotation, also locally preserve angles and are called *conformal.*

Conformal group in d>2

Definition: $g'_{\mu\nu}(\vec{x}') = \Lambda(\vec{x}) g_{\mu\nu}(\vec{x})$

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$$

$$x'^\mu = x^\mu + a^\mu$$ translation $P_\mu = -i\partial_\mu$

$$x'^\mu = \alpha x^\mu$$ dilation $D = -ix^\mu \partial_\mu$

$$x'^\mu = M^\mu{}_\nu x^\nu$$ rotation $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

$$x'^\mu = \frac{x^\mu - b^\mu \vec{x}^2}{1 - 2\vec{b}\cdot\vec{x} + b^2 \vec{x}^2}$$ SCT $K_\mu = -i(2x_\mu x^\nu \partial_n u - \vec{x}^2 \partial_\mu)$

$$SO(d+1,1)$$

Conformal invariance in d=2

In d=2 there is an infinite number of coordinate transformations that are locally conformal: those are conformal mappings of the plane onto itself

The notion of primary field:

$$\phi'(w,\overline{w}) = (\frac{dw}{dz})^{-h}(\frac{d\overline{w}}{d\overline{z}})^{-\overline{h}}\phi(z,\overline{z})$$

(quasi-primary fields – previous is valid only under global conformal group)

2-point and 3-point correlation functions are fixed by conformal invariance

Role of the stress tensor

$$x^\mu \to x^\mu + \epsilon^\mu$$

$$\delta S = \int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu$$

Invariance under:

-translation → conserved

-rotation → symmetric

-scale transformation → traceless

$$\left.\begin{array}{l} \\ \end{array}\right\} \begin{array}{l} T_{z\bar{z}} + T_{\bar{z}z} = 0 \\ T_{z\bar{z}} = T_{\bar{z}z} = 0 \end{array}$$

The only nonzero components are

$$T_{zz} = T(z), T_{\overline{zz}} = \overline{T}(\overline{z})$$

Example: free scalar field

$$S = \frac{g}{٢}\int d^{٢}r(\partial_\mu\phi)(\partial^\mu\phi) \propto \int(\partial_z\phi)(\partial_{\bar{z}}\phi)d^{٢}z$$

Correlation function and OPE:

$$< \phi(x)\phi(y) >= -\frac{1}{4\pi g}ln(x-y)^2$$

$$\partial\phi(z)\partial\phi(w) \sim -\frac{1}{4\pi g}\frac{1}{(z-w)^2}$$

$$T_{\mu\nu} = g(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi)$$

$$T(z) = -2\pi g : \partial\phi\partial\phi :$$

This OPE shows that ϕ is a primary field

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{(z-w)}$$

But T is not primary!

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

Conformal Ward identity

Like in any QFT, they express consequence of the symmetry of the action (and integration measure) on correlation functions

$$\delta_{\epsilon,\epsilon'} < \phi_1(z_1)\phi_2(z_2)... >=$$
$$-\frac{1}{2\pi i}\int dz \epsilon(z) < T(z)\phi_1(z_1)\phi_2(z_2)... > + \text{c.c.}$$

$$T(z)\phi(w) \sim \frac{h}{(z-w)^2}\phi(w) + \frac{1}{z-w}\partial_w\phi(w)$$

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

conformal anomaly, central charge

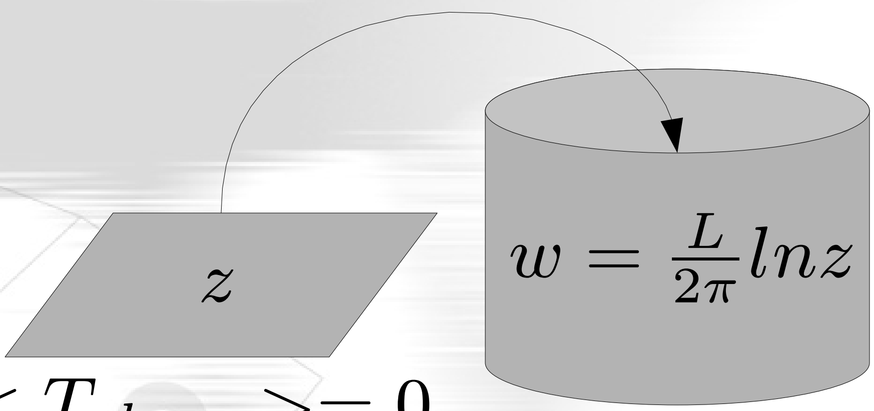
Determines the theory, together with scaling dimension and OPE coefficients

Transformation of stress tensor

$$T'(w) = (\frac{dw}{dz})^{-2}(T(z) - \frac{c}{12}\{w;z\})$$

$$\{w;z\} = \frac{(d^3w/dz^3)}{(dw/dz)} - \frac{3}{2}(\frac{d^2w/dz^2}{dw/dz})^2$$ Schwartzian

Physical meaning of central charge



$$< T_{plane} >= 0$$

=describes how system reacts to the introduction of macroscopic length scale that breaks conformal invariance

$$< T_{cyl}(w) >= -\frac{c\pi^2}{6L^2}$$

Radial quantization

Massless CFT, must quantize on a circle of fixed radius in ordinary QFT, Hamiltonian is the integral over the appropriate component of stress tensor, here one takes instead the generator of scale transformations:

$|\phi >= \lim_{z\to 0} \phi(z,\overline{z})|0>$ asymptotic state

$[\phi(z,\overline{z})]^\dagger = \overline{z}^{-2h} z^{-2\overline{h}} \phi(1/\overline{z}, 1/z)$ Hermitian conj

$$D = L_0 + \overline{L}_0 = \frac{1}{2\pi i}\int zT(z)dz + c.c.$$

$$\phi(z) = \sum_m z^{-m-h}\phi_m, \phi_m = \frac{1}{2\pi i}\int dz z^{m+h-1}\phi(z)$$

$$\mathcal{T} \to \mathcal{R}$$

$$[A,B] = \int_0 dw \int_w dz a(z) b(w)$$

$$(A = \int a(z)dz, B = \int b(z)dz)$$

Virasoro algebra

$$Q_\epsilon = \frac{1}{2\pi i}\int dz\, \epsilon(z) T(z) \xrightarrow{\text{Ward}} \delta_\epsilon \Phi = -[Q, \Phi_w]$$

↑ Conformal charge!

Expand $T(z), \epsilon(z)$ in modes;

L_n and L_n are generators of local conformal transf on the Hilbert space. In particular, L_{-1}, L_0, L_1 are generators of global conformal transformations.

Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)$$

same as in Witt algebra (classical)

central term

The Hilbert space

Define vacuum as: $L_n|0> = \overline{L}_n|0> = 0, n \geq -1$

Asymptotic state $|h> = \phi(0,0)|0>$

is an eigenstate of $L_0|h> = h|h>$

and annihilated by all $L_n|h> = 0, n > 0$

Moreover, $[L_0, L_{-m}] = mL_{-m}$

so the excited states are obtained recursively via ladder operators:a

$L_{-k_1}...L_{-k_n}|h>$ descendants

Such a subspace forms the representation of Virasoro algebra (so-called Verma module)

Structure of Hilbert space

$$\left.\begin{array}{l} \ldots \\ L_{-2}|\phi>, L^2_{-1}|\phi> \\ L_{-1}|\phi> \\ |\phi> \end{array}\right\}$$ Is this rep unitary? Decomposable?

Decomposability implies the existence of null states:

$$L_n(L_2|\phi> -(1/g)L^2_{-1}|\phi>) = 0$$

$$\Delta = \frac{3g-2}{4}, c = \frac{(3g-2)(3-2g)}{g}$$

special case of $(r,s) = (2,1)$ Katz's formula:

$$\Delta_{r,s}(g) = \frac{(rg-s)^2-(g-1)^2}{4g}$$

If c, Δ are such, then $|\phi>$ has null state at level $r \cdot s$

Fusion rules, Kac table, unitarity

$$< \phi_{2,1}(z_1)\phi_{r,s}(z_2)\phi_{\Delta}(z_3) > \longrightarrow \Delta = \Delta_{r\pm 1,s}$$

$g = p/p'$ (rational)

$$\Delta_{r,s} = \Delta_{p'-r,p-s}$$

Same primary field sits at two different positions in the Kac table $1 \leq r \leq p' - 1, 1 \leq s \leq p - 1$

Fusion algebra is truncated, fields inside rectangle do not couple to those outside – finite number of fields (defines minimal models)

Example: The Ising model

$$< \sigma_i \sigma_{i+n} >= |n|^{-1/4}$$

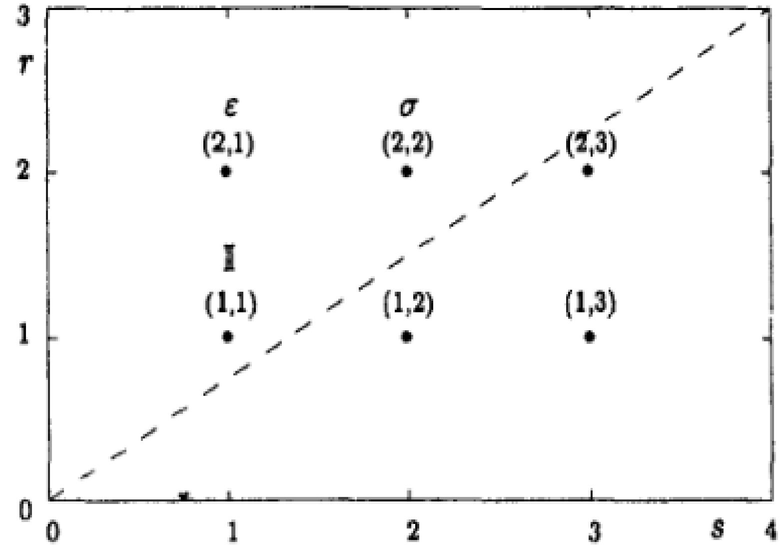
$$< \epsilon_i \epsilon_{i+n} >= |n|^{-2}$$

Operator content:

$$I \equiv \phi_{1,1}, \phi_{2,3}$$

$$\sigma \equiv \phi_{2,2}, \phi_{1,2}$$

$$\epsilon \equiv \phi_{2,1}, \phi_{1,3}$$



$$c = \frac{1}{2}, \mathcal{M}(4,3)$$

Fusion rules:

$$\sigma \times \sigma = I + \epsilon$$

$$\sigma \times \epsilon = \sigma$$

$$\epsilon \times \epsilon = I$$

CFT – quantum Hall connection

Early work of Witten established connection between Hilbert space of (2+1) CS theory and the space of corr functions of certain (1+1) CFTs

Seminal paper of Moore and Read

In reality, it is safest to think of CFT in quantum Hall physics as a tool to propose ground state wavefunctions for the bulk (and one can also describe the edge in the same manner)

Vertex operators $V_\gamma(z) =: e^{i\gamma\phi(z)}:$

$$< V_{\gamma_1}(z_1) ... V_{\gamma_n}(z_n) >= \prod_{i<j} |z_i - z_j|^{4\gamma_i\gamma_j}$$

$$\sum_i \gamma_i = 0$$

Laughlin wf, quasihole, statistics

$$< V_{\sqrt{m}}(z_1)\ldots V_{\sqrt{m}}(z_n)\mathcal{O}_{bg} >= \prod_{i<j}(z_i - z_j)^m e^{-1/4\sum_i |z_i|^2}$$

$$V_{qh}(w) = e^{\frac{i}{\sqrt{m}}\phi(w)} \xrightarrow{\text{insertion}} \times \prod_i (z_i - w)$$

Depletes the electron liquid and from the way the electrons are driven away, it turns out that the charge of quasihole is $e^* = e/m$

2 quasiholes $\sim (w_1 - w_2)^{1/m}$ monodromy

Exchanging the quasiholes, one gets the phase

$$\Psi(w_1, w_2) = \exp(i\theta)\Psi(w_2, w_1), \theta = \pm\pi/m$$ fractional statistics

Exotic states: Moore-Read Pfaffian

$$\Psi_\alpha(w_i, w_j) = M_{\alpha\beta}\Psi_\beta(w_j, w_i)$$

↑ some unitary matrix

Requires a multiplet of degenerate states
 - encoded in fusion rules:

$$\phi_a \times \phi_b = \phi_c$$ Abelian

$$\phi_a \times \phi_b = \sum_c N_{abc}\phi_c$$ non-Abelian

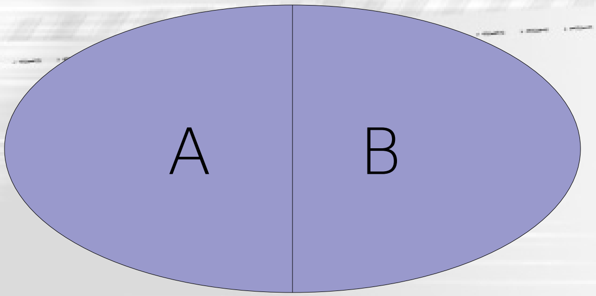
$$V_{el}(z) = \Psi V_{\sqrt{m}}(z)$$

$$< V_{el}(z_1)...V_{el}(z_n)\mathcal{O}_{bg} >= Pf(\frac{1}{z_i - z_j}) \prod_{i<j}(z_i - z_j)^m$$

$$\mathcal{A}(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4} + ...)$$

Topological entanglement entropy

A way to detect the signature of underlying CFT



Schmidt decomposition

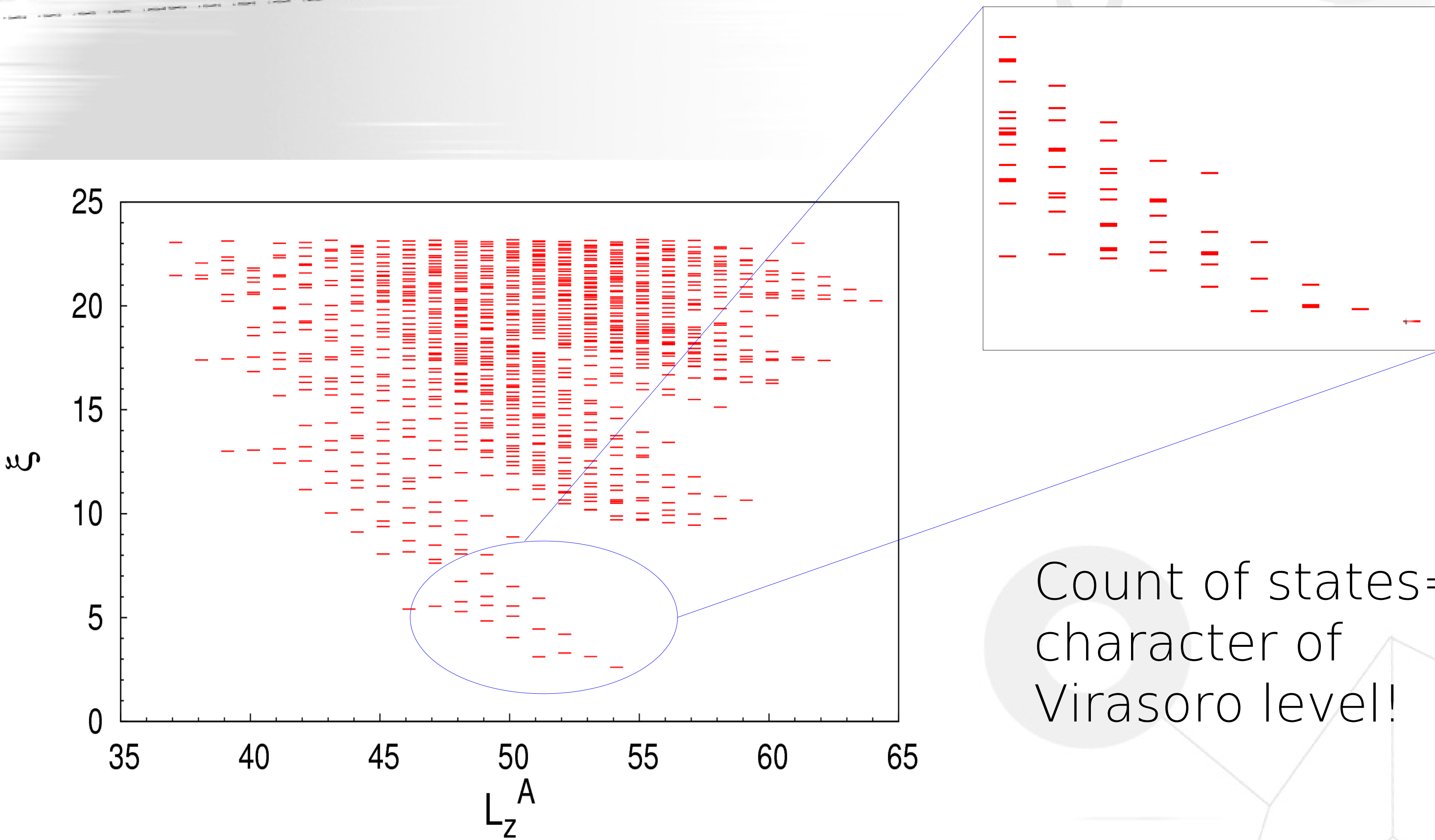
$$|\Psi >= \sum_i e^{-\xi_i/٢} |\Psi_A^i > \otimes |\Psi_B^i >$$

Entanglement entropy = von Neumann entropy of the block

$$\rho^L = \frac{\sum_\lambda e^{-\xi_\lambda} |\Psi_\lambda^L >< \Psi_\lambda^L|}{\sum_\lambda e^{-\xi_\lambda}}$$

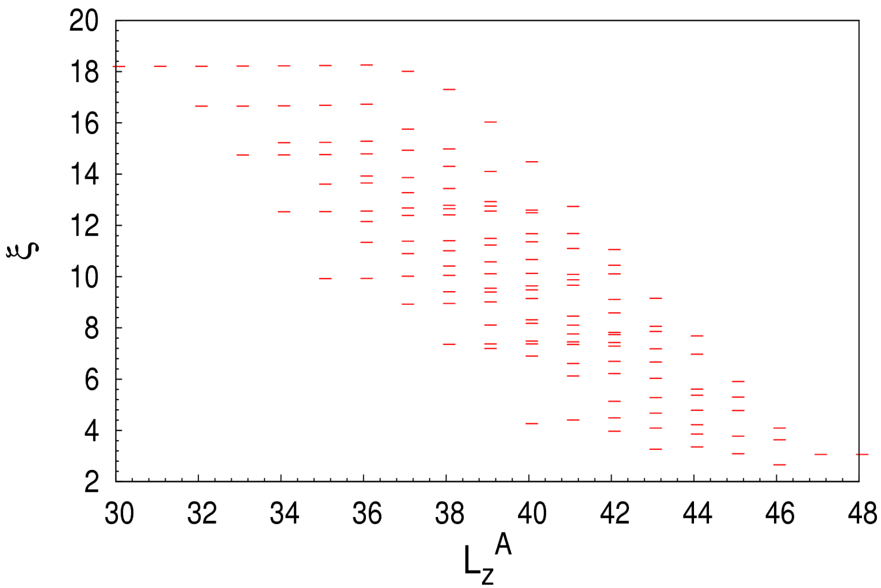
$$S = -Tr\{\rho \log \rho\}$$

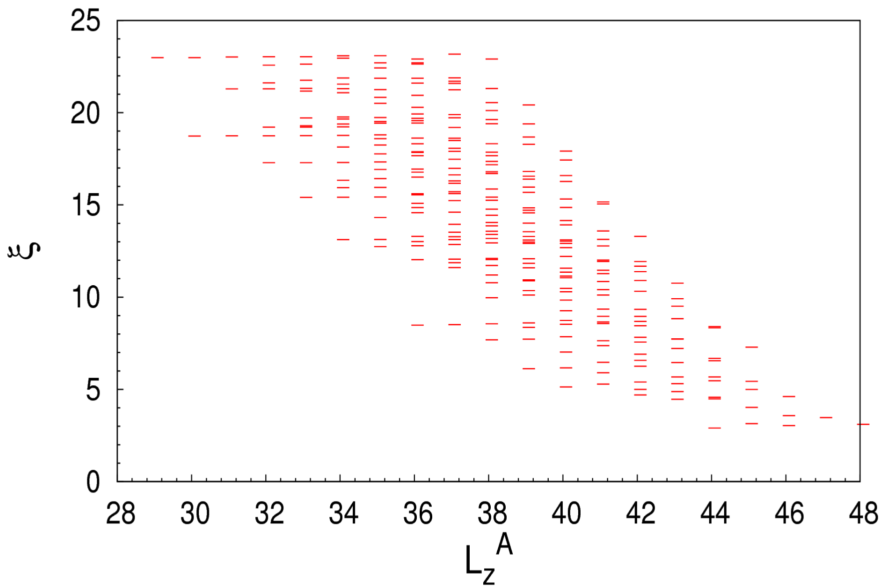
Entanglement spectrum for Laughlin

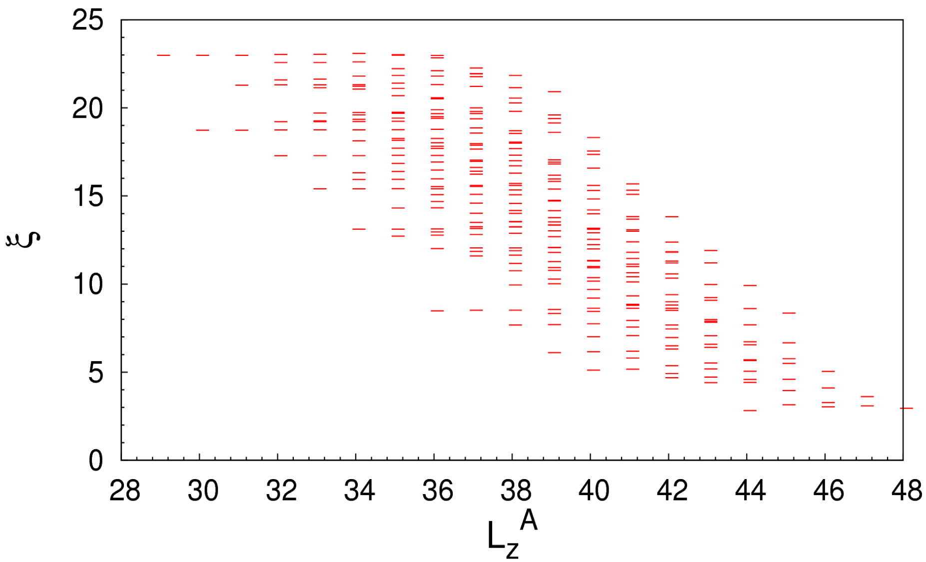


Count of states=
character of
Virasoro level!

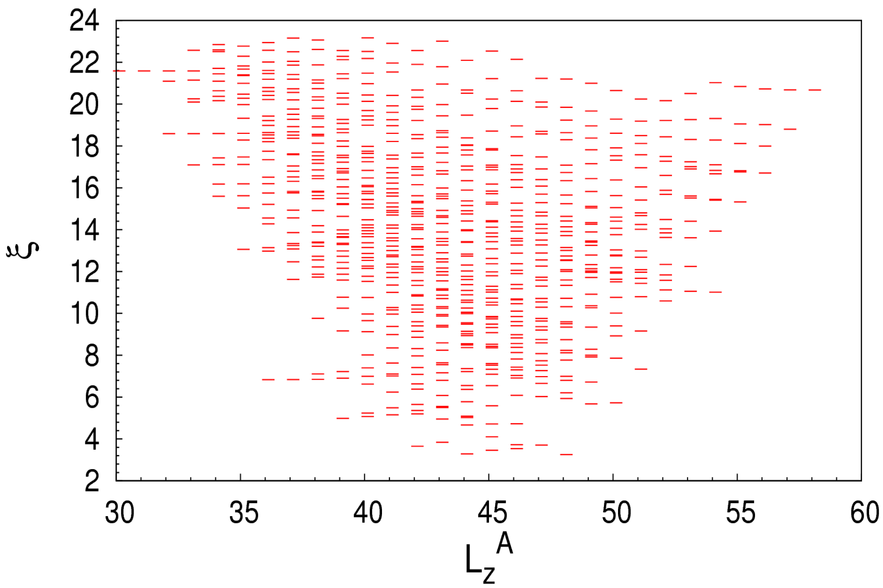
Moore-Read state

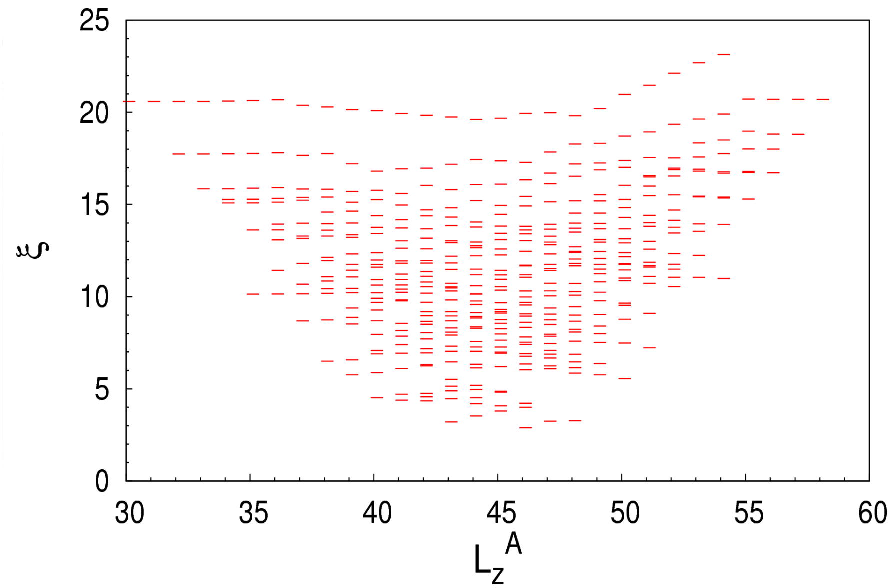


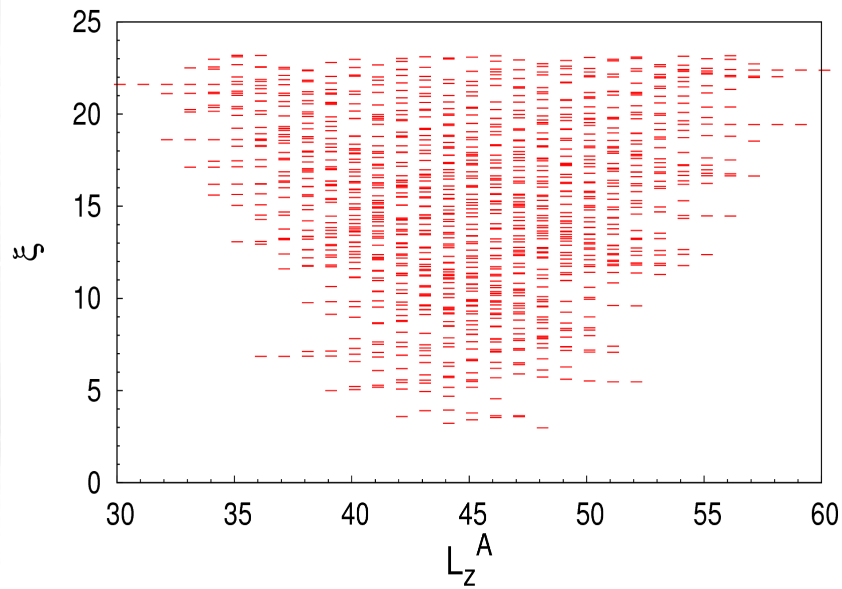




Coulomb state at Pfaffian shift







Conclusions

CFT is a very powerful tool for studies of critical systems in 2d

It can be a very useful tool for systems that exhibit quantum Hall effect

But the connection between QHE and CFT is not, from physical point of view, as firmly justified, therefore additional considerations (like entanglement spectrum) are needed to complement and understand the real nature of this connection